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A BOUNDARY-VALUE PROBLEM ON DRAINAGE AT THE EDGE OF FRESH GROUND WATER AND ITS APPLICATIONS†

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An algorithm for calculating the geometrical characteristics of an edge, including the position of its free boundaries, has been developed and implemented as a computer program based on a previously obtained [1, 2] analytic solution of the problem of filtration at the edge of drainage of fresh ground water which is above saline water. A straightforward sequence of calculations, with a preliminary determination of the conformal mappings, enabled a complete hydrodynamic analysis to be made of the flow at the edge with the interpretation of the drain as a water supply. Here, in each version of the numerical calculations, the optimal depth for laying the water supply is established which ensures its maximum possible productivity on the brink of destabilization of both free boundaries of the edge. Another application of this boundary-value problem is the calculation of the zone of demineralization of the soil layer which contained saline ground waters as a result of their replacement by fresh surface waters and drainage. Copyright © 1996 Elsevier Science Ltd.

1. THE BOUNDARY-VALUE PROBLEM AND THE PROBLEM OF THE PARAMETERS

We shall consider the free steady-state filtration of fresh water above quiescent saline waters from periodically arranged linear surface sources to point drains of equal capacity laid halfway between them at the same depth. The half period of such a fresh water edge is depicted schematically in Fig. 1(a) as the flow domain.

We will formulate the boundary-value problem of finding the complex stream potential relative to the filtration coefficient $\omega(z) = \varphi + i\psi$ (φ is the filtration velocity potential and ψ is the stream function [3]) which is an analytic function of the complex coordinate $z = x + iy$ of the points of the flow domain, subject to the following boundary conditions, which are matched with the choice of the system of coordinates

$$GE\colon\ y=0,\ \varphi=0;\quad AB\colon\ x=0,\ \psi=G;\quad BC\colon\ x=0,\ \psi=0$$

$$ED\colon\ x=L,\ \psi=0;\quad AG\colon\ \varphi-y=0,\ \psi=Q;\quad CD\colon\ \varphi+\rho y=C,\ \psi=0 \tag{1.1}$$

Here, Q is the discharge of filtered water (from the point drain B) in a calculation on a chosen flow domain and $\rho = \rho_2/\rho_1 - 1$ (ρ_1 and ρ_2 are the densities of the fresh and saline waters). The first condition on the segment CD, which is specific to this problem, is based on prerequisites concerning the immobility of the saline water and on the continuity of the pressure on crossing the line separating this water from the filtering fresh water [4].

The problem in question arose as an extension of the problem of the drainage in a soil layer with a confining stratum. The construction of the solution for this scheme with its subsequent extension to the problem under consideration has been described previously [2]. It is based on conformal mappings of the domain of the function ω (Fig. 1b) and the two-sheeted domain $1/w$ (Fig. 1c) of inversion of the hodograph of the filtration velocity $\bar{w} = w_x + iw_y$, which is conjugate with the complex velocity $w = w_x - iw_y = d\omega/dz$ [3] in the half-plane $\operatorname{Im}\zeta \geq 0$ (Fig. 1d). On the segment $0 \leq \zeta \leq g$, which corresponds to the depression curve, the mapping functions can be represented in the form

$$\omega(\zeta) = \frac{Q\sqrt{(1+b)(b+g)}}{\pi}\int_{\zeta}^{g}\frac{du}{(b+u)\sqrt{(1-u)(g-u)}} + iQ =$$

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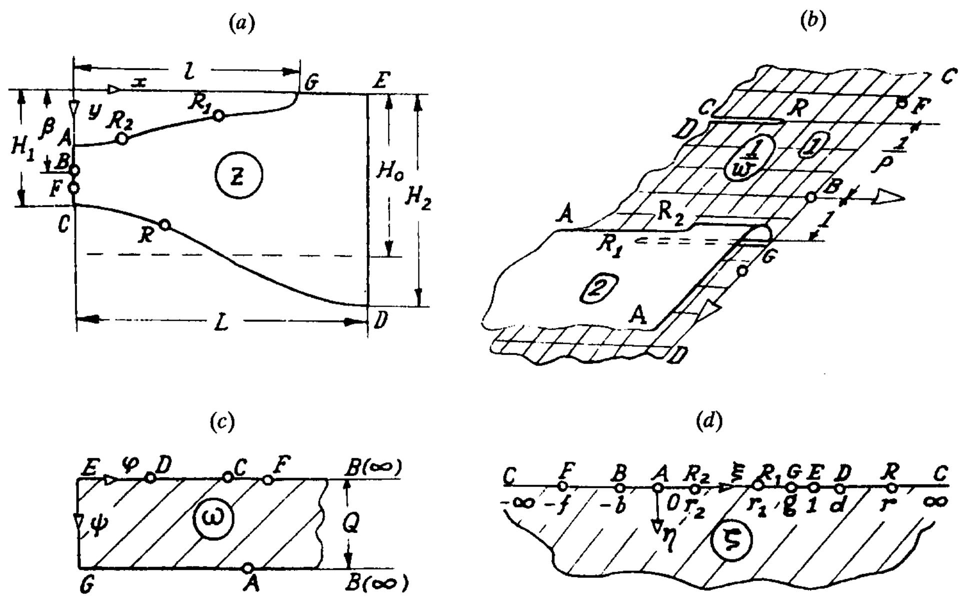

Fig. 1.

$$= \frac{2Q}{\pi} \operatorname{arsh} \sqrt{\frac{(1+b)(g-\zeta)}{(1-g)(b+\zeta)}} + iQ \tag{1.2}$$

$$\frac{1}{w(\zeta)} = \frac{dz}{d\omega} = \frac{M}{2} \int_{\zeta}^{g} \frac{P(u)du}{\sqrt{u^3(g-u)(d-u)^3}} + i =$$

$$= -M \frac{(f+\zeta)\sqrt{g-\zeta}}{\sqrt{\zeta(d-\zeta)}} - \left(1+\frac{1}{\rho}\right)\frac{2K}{\pi} Z(\alpha, k) + i; \quad M > 0$$

Here

$$P(u) = u^3 - c_1 u^2 + c_2 u - c_3; \quad c_1 = 2d + f + aE$$

$$c_2 = g(d + 2f + a\Phi), \quad c_3 = dfg; \quad a = \left(1 + \frac{1}{\rho}\right) \frac{2\sqrt{d}}{\pi M}$$

$$Z(\alpha, k) = E(\alpha, k) - \frac{E}{K} F(\alpha, k), \quad \Phi = \frac{E - k'^2 K}{k^2} \tag{1.3}$$

$$\alpha = \arcsin \sqrt{(1 - \zeta / g) / (1 - \zeta / d)}, \quad k = \sqrt{g / d}, \quad k' = \sqrt{1 - k^2}$$

Incomplete $F(\alpha, k)$, $E(\alpha, k)$ and complete K, E elliptic integrals of the first and second kinds with modulus k in the normal Legendre formula as well as the Jacobian zeta-function $Z(\alpha, k)$ associated with it [5, formulae (110.02), (110.03) and (140.01)] occur in these expressions.

Using relation (1.2) for $1/w$, transformed to the segment AB ($-\infty < \zeta < 0$), and the condition $(1/w)_{\zeta = -b} = 0$, we have

$$M = \frac{\sqrt{b(b+d)}}{\rho(f-b)\sqrt{b+g}} J(\rho, b, d, g); \quad J = 1 - (1+\rho)\Lambda_0\left(\arcsin\sqrt{\frac{d}{b+d}}, k\right) \tag{1.4}$$

This contains another, also standardized, elliptic lambda-function, the Heuman function (C. Heuman [5, formula (150.03)])

$$\Lambda_0(\gamma,k)=\frac{2}{\pi}[EF(\gamma,k')+KE(\gamma,k')-KF(\gamma,k)] \tag{1.5}$$

Since $M > 0$, it then follows from (1.4) that

$$f<b \text{ when } J<0;\quad f>b \text{ when } J>0 \tag{1.6}$$

The limiting case when $f = 0$ is possible within the framework of the first relation. In this case, the hydrodynamic pressure in the segment AB, $p \leqslant 0$ and the equality is only satisfied at the point A, which becomes the cusp of the depression curve [6]. Subsequent reduction of the pressure in the segment indicated by as small an amount as desired while increasing the withdrawal of water destroys the dynamic equilibrium between the flow and the air phase above it, which leads to the break through of air into the water supply.

The limiting case when $f = \infty$, which is characterized by the relation $dp/dy \geqslant \rho_2 g$ in the segment BC, is associated with the second relation of (1.6). The equality is only preserved at point C, which is converted into the cusp of the line of separation [1]. Dynamic equilibrium between the fresh water, which is moving at the edge, and the saline water lying under it is maintained, descriptively speaking, only at this point as, within the limits of the segment BC, the hydrodynamic pressure gradient already exceeds the hydrostatic equilibrium gradient in the saline water zone and the further activation of the water supply by as small an amount as desired leads to a break through of the saline water into it.

From relations (1.2), we have after some reduction which includes integration by parts [2, Section 12]

$$z(\zeta)=z(\zeta_0)+\frac{\omega(\zeta)-\omega(\zeta_0)}{w(\zeta_0)}+\int_{\zeta_0}^{\zeta}\frac{d}{du}\left(\frac{1}{w(u)}\right)[\omega(\zeta)-\omega(u)]du \tag{1.7}$$

In connection with the segment AG, we take $\zeta_0 = g$, $z(\zeta_0) = l$, $\omega(\zeta_0) = iQ$, $w(\zeta_0) = -i$ in (1.7) (Fig. 1). Using the integral representation (1.2) for the function $1/w(\zeta)$, we obtain the equation of the depression curve in the complex parametric form $(0 \leqslant \zeta \leqslant g)$

$$z(\zeta)=l+Q+i\omega(\zeta)+\frac{M}{2}\int_{\zeta}^{g}\frac{P(u)[\omega(\zeta)-\omega(u)]du}{\sqrt{u^3(g-u)(d-u)^3}}\quad (0\leqslant\zeta\leqslant g) \tag{1.8}$$

The function $\omega(\zeta)$ is defined by the first relation of (1.2).

Guided by the direct formulation of the problem, we agree to specify, in each actual version, the values of Q and ρ and the geometrical parameters of the edge l, L, β and H_0. Using these parameters, we derive the following system of equations in the four unknown parameters of the conformal mappings b, d, g and f

$$\frac{M}{2}\int_0^g U(u)[\varphi(u)-\varphi(0)]du=l\quad\left(\varphi(u)=\frac{2Q}{\pi}\operatorname{arsh}\sqrt{\frac{(1+b)(g-u)}{(1-g)(b+u)}}\right)$$

$$l+Q-\frac{MQ}{\pi}\int_g^1 U(u)\arcsin\sqrt{\frac{(b+g)(1-u)}{(1-g)(b+u)}}du=L$$

$$\varphi(0)+\frac{M}{2}\int_{-b}^{0}U(u)[\varphi(0)-\varphi(u)]du=\beta \tag{1.9}$$

$$\frac{Q\sqrt{(1+b)(b+g)}}{\pi L}\int_d^\infty\left[H_2+\frac{\varphi_1(d)-\varphi_1(u)}{\rho}\right]\frac{|\operatorname{Re}(1/w)_{CD}|\,du}{(b+u)\sqrt{(u-1)(u-g)}}=H_0$$

$$U(u)=\frac{P(u)}{\sqrt{|u^3(g-u)(d-u)^3|}},\quad \varphi_1(u)=\frac{2Q}{\pi}\operatorname{arsh}\sqrt{\frac{(b+g)(u-1)}{(1-g)(b+u)}},\quad \operatorname{Re}\left(\frac{1}{w}\right)_{CD}=$$

$$= -\left\{\left[M(f+u)+\left(1+\frac{1}{\rho}\right)\frac{2K}{\pi}\sqrt{d}\right]\sqrt{\frac{u-g}{u(u-d)}}-\left(1+\frac{1}{\rho}\right)\frac{2K}{\pi}Z\left(\arcsin\sqrt{\frac{d}{u}},k\right)\right\}$$

The first equation of the system follows from equality (1.8) when $\zeta = 0$, $z = iy_A$, $\omega = \varphi(0) + iQ$, taking into account the relation $y_A = y(0) = \varphi(0)$, which is defined by the first boundary condition in (1.1) on the depression curve AG. In deriving the two following equations, which include the quantities L and β, dependence (1.7) for $z(\zeta)$ was first transformed to fit segments GE and AC. The final equation reflects the condition that the initial volume of the saline water is conserved during the formation of the edge. In the initial notation, this condition leads to the relations [1]

$$\int_0^L y_{CD}(x)dx = -\int_d^\infty y_{CD}(u)\frac{dx_{CD}(u)}{du}du = H_0 L \tag{1.10}$$

The dependence $y_{CD}(u)$ is obtained from the first condition (1.1) in CD: using point D, it can be written in the form

$$\varphi_1 + \rho y = \varphi_1(d) + \rho H_2$$

The function $\varphi_1(u)$ is contained in (1.9). As far as the maximum depth of the edge H_2 (Fig. 1a) is concerned, it is calculated with the remaining constants occurring in the integral before each conversion to the integral on the left-hand side of the fourth equation of (1.9). The expression for H_2 follows from formula (1.7), transformed to segment ED when $\zeta_0 = 1$.

Returning to the integrand in the second integral of (1.10), we again make use of boundary conditions (1.1) in the segment CD and the integral representation for $\omega_{CD}(u) = \varphi_{CD}(u)$ $(d \leqslant u \leqslant \infty)$, which is obtained from the first dependence of (1.2), In this case, we have

$$\frac{dx_{CD}(u)}{du} = \frac{dz}{du} - i\frac{dy}{du} = \frac{dz}{d\omega}\frac{d\varphi}{du} + \frac{i}{\rho}\frac{d\varphi}{du} = \frac{Q\sqrt{(1+b)(b+g)}}{\pi(b+u)\sqrt{(u-1)(u-g)}}\operatorname{Re}\left(\frac{1}{w}\right)_{CD}$$

The notation for the expression Re $(1/w)_{CD}$ occurring in (1.9) is based on representation (1.2) for the function $1/w(\zeta)$, which has been transformed to fit the segment CD ([2], formula (13.3)).

The values of l, L, β and H_0 are specified taking account of the natural constraints $0 < l < L$, $0 < \beta < H_0$. The magnitude of Q must not exceed the value at which one or other of the two above-mentioned critical flow conditions occurs. It is precisely for such a condition that it is necessary to carry out a preliminary calculation of the flow at the edge. However, it is not known in advance how it will turn out to be in the case of a chosen combination of the above-mentioned physical parameters.

2. DOUBLE CRITICAL BEHAVIOUR

A singular situation, which is inherent in the problem under consideration, of the flow on the brink of the destabilization of the two free boundaries with the simultaneous conversion of their points A and C into cusps (Fig. 2a), has been pointed out previously [2] as the key stage in the investigation. In the case of such flow behaviour, which will henceforth be called double critical flow, the whole of the second sheet of the domain of $1/w$ (Fig. 1c) degenerates into the point $1/w = i$ and the quadrant Re $(1/w) \leqslant 0$, Im $(1/w) \leqslant 0$ degenerates into the point $1/w = -i\rho$ (Fig. 2b). A conformal mapping of the domain in the half-plane Im $\zeta \geqslant 0$ (Fig. 2c) is carried out by means of the function

$$\frac{1}{w} = N\int_\zeta^g \frac{(r-u)du}{\sqrt{u(g-u)(d-u)^3}} + i = -\left(1+\frac{1}{\rho}\right)\frac{2}{\pi}KZ(\alpha,k) + i \tag{2.1}$$

Relation (1.2) for the function $w(\zeta)$ is preserved, as is representation (1.7) for the function $z(\zeta)$ in which we now have, starting from (2.1) and using the formula for the differentiation of the function $Z(\alpha, k)$ with respect to an argument [4, formula (730.03)]

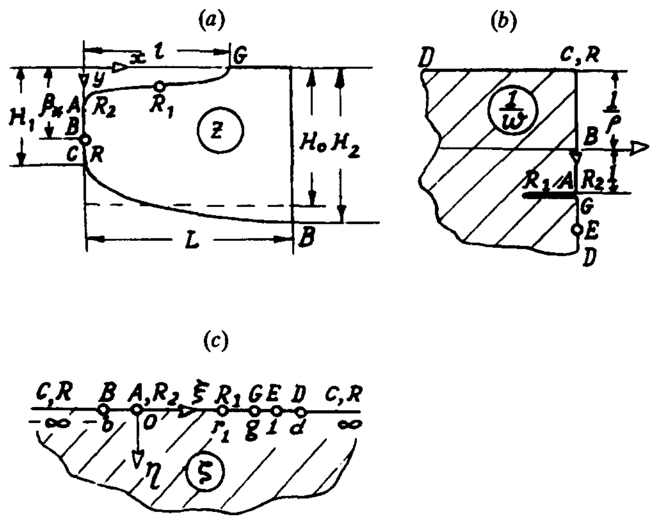

Fig. 2.

$$\frac{d}{du}\left(\frac{1}{w}\right)=N\frac{u-r}{\sqrt{u(g-u)(d-u)^3}};\quad N=\left(1+\frac{1}{\rho}\right)\frac{E\sqrt{d}}{\pi};\quad r=\frac{g\Phi}{E} \tag{2.2}$$

The function Φ is defined in the group of formulae (1.3).

In the case being considered, the solution of the problem therefore contains three unknown mapping parameters: b, d and g. A relation exists between them which is expressed by the condition $(1/w)_{\zeta=-b} = 0$. Using the representation of the function $1/w(\zeta)$ in the segment AC which is obtained from dependence (2.1), this condition leads to the relation (cf. (1.4))

$$J(\rho,b,d,g)=1-(1+\rho)\Lambda_0(\arcsin\sqrt{d/(b+d)},\sqrt{g/d})=0 \tag{2.3}$$

The function Λ_0 is expressed by formula (1.5).

So, in the case when there are three defining geometrical characteristics of the scheme (l, L and H_0), just two of the mapping parameters are now independent: we shall assign the parameters d and g to them. The filtration rate Q from the drain B, together with these parameters is also subject to calculation from the system of three equations associated with the above-mentioned characteristics. These equations are analogous to the first, second and fourth equations of system (1.9), and an analogue of the third equation of (1.9) is then used to find the ordinate $\beta = \beta_*$ of the drain. On eliminating the quantity Q from this system, we obtain the system

$$f_1(d,g)=l/L,\quad f_2(d,g)=L/H_0 \tag{2.4}$$

The solution of the initial system is thereby reduced at the first stage to finding the parameters d and g, where

$$g\in(g_0,1),\quad d\in(1,d_0);\quad g_0=\mathrm{sn}^2(lK,k),\quad d_0=1/k^2 \tag{2.5}$$

Here, sn is the Jacobi elliptic function ([5], formula (120.010).

Equalities (2.5) determine the affices of the points G and D in the ζ-plane when $Q = 0$ and the domain of z is a rectangle. Constraints (2.5) are associated with the following tendency to stability which is revealed and made use of in the calculations: when the filtration rate Q increases, when the values of the other physical input parameters are fixed, the parameters g and d approach one another and unity by increasing and decreasing respectively (Fig. 1d). This agrees with the change in the relative arrangement of the singular corner points on the boundary of the flow domain in the case of an intensification of the drain as a result of which the non-stationary points $A(0)$ and $C(\infty)$ approach it while removing, at the same time, the first of these points from the fixed points $G(g)$ and $E(1)$, and the second of these points from the points $E(1)$ and $D(d)$ (Fig. 1a, d). It is true that the last point is movable in the domain of z but, nevertheless, the nature of the dependence of its affix d on the rate Q which has been noted above is unfailingly exhibited in the calculations.

We now represent the second equation of system (2.4) in the form

$$F_2(d) = f_2[d, g(d)] = L / H_0 \tag{2.6}$$

Equation (2.6) is solved for the parameter d, and the parameter g is determined from the first equation of (2.4) in an internal cycle for each of the values of d which are touched upon the solution process. Here, for each pair of parameters d and g, the determination of the value of b from relation (2.3) precedes the calculation of the function $f_1(d, g)$, which also includes the parameter b. The implementation of such procedures is based on a detailed numerical investigation of the behaviour of each of the functions in the equations and on the identification of the integrals in which the required parameters must be contained. In particular, the parameter g is determined from the first equation of the system taking account of the monotonic growth in the function $f_1(g, d)$ which has been established when the parameter g is increased (in this case, as was mentioned above, the parameter d is fixed). On the other hand, the function $F_2(d)$ in Eq. (2.6) is a monotonically decreasing function.

Having found the parameters d and g in the double critical state, the filtration rate $Q = Q_*$ of the drain is determined from an equation associated with the magnitude of H_0. The ordinate of the drain $\beta = \beta_*$ is then calculated. Actually, the value of Q_* turns out to be the maximum attainable productivity of the water supply from the edge which is ensured when the drain is located at a depth β_*. It is as though the double critical state borders all the drainage states which can be obtained in a problem for a specified combination of physical input parameters ρ, l, L and H_0 when the ordinate β of the drain B is varied in the range $(0, H_0)$, which also includes the simple critical states mentioned above, associated with any one of the free boundaries.

3. FLOW IN A WATER SUPPLY

We will now consider the problem of the recovery of fresh ground water from an edge. The boundary-value problem formulated in Section 2 is directed towards solving this.

By finding the value of β_* we can choose one of the two simple critical states into which the filtration process passes when the intensity of the drain, laid at a depth β, is increased: $f = 0$ when $\beta < \beta_*$ or $f = \infty$ when $\beta > \beta_*$. All relations, equations and integrands are obtained for these conditions from those presented above by taking the limit with respect to the parameter f.

System (1.9) is used to find the parameters b, d and g, which are now subject to the constraints

$$b \in (0, b_*) \text{ when } \beta < \beta_*, \quad b \in (b_*, \infty) \text{ when } \beta > \beta_*; \quad g \in (g_0, g_*), d \in (d_*, d_0) \tag{3.1}$$

Values of parameters calculated in the double critical state are given an asterisk.

Furthermore, the discharge Q, which was initially also eliminated from system (1.9), is here also subject to determination by means of its transformation to the form (cf. (2.4), (2.6))

$$\begin{aligned} F_1(g) &= f_1(b, d, g) = l / L \\ F_2(d) &= f_2[b, d, g(b, d)] = L / H_0 \\ F_3(b) &= f_3[b, d(b), \ g(b, d)] = \beta / l \end{aligned} \tag{3.2}$$

In accordance with this notation, the function f_3 in the third equation of system (3.2), which is solved in the external cycle of the procedure, is represented as a complex function of the parameter b, the change in which in some of the simple critical states is regulated by constraints (3.1). For each of the values of the parameter b, which are touched upon during the course of the solution and are fixed during conversion to the first two equations of the system, the parameters d and g are found using the scheme described above, and these parameters are also introduced into the function $F_3(b)$ which has to be calculated at each step of the above-mentioned external cycle.

Comparing the last two constraints of (3.1) with the corresponding constraints in (2.5), we see that the calculation of the double critical state enables one to narrow down the initial search intervals for the parameters d and g by matching them with the range $(0, Q_*)$ of possible values of the discharge Q. Here and in the subsequent calculations, observation of these corrected intervals in conjunction with the corresponding constraint on the change in the parameter b ensures that the condition $M > 0$ is satisfied and also ensures a monotonic increase in the function $F_3(b)$ from 0 to β_*/l when $f = 0$ and from β_*/l to H_0/l when $f = \infty$ which is used in solving the third equation of system (3.2).

After the parameters b, d and g have been found from the fourth equation of system (1.9), the discharge Q is calculated as before. We denote the value of the discharge when $f = 0$ and $f = \infty$ by Q_1 and Q_2, respectively. The dependence of Q on the depth of the drain when $l = 40$, $L = 50$, $H_0 = 20$, $\rho = 0.02$ is illustrated by the graph in Fig. 3. The coordinates $\beta = \beta_* = 0.993$ and $Q = Q_* = 0.180$ of its maximum point are found in the double critical state. As a result, the question of the permissible values of Q when the drain is laid at an arbitrary depth β is clarified, which provides a means of calculating the flow for any value of Q which does not pass beyond the boundary established for it.

In the case of such drainage conditions, which are henceforth referred to as normal conditions, the system of Eqs (1.9), as has already been mentioned, is subject to solution in the whole volume. We transform it to the form (cf. (3.2))

$$
\begin{aligned}
&F_1(g) = f_1(b, f, d, g) = l / L \\
&F_2(d) = f_2[b, f, d, g(b, f, d)] = L / H_0 \\
&F_3(b) = f_3[b, f(b), d(b, f),\ g(b, f, d)] = \beta / l \\
&F_4(f) = f_4[b, f, d(b, f), g(b, f, d)] = Q / H_0
\end{aligned}
\tag{3.3}
$$

The four-layer cyclic procedure for solving the system of Eqs (1.9), which is implemented in the computer program, is reflected in this notation. The third equation of system (3.3) is solved for the parameter b, as in the case of a simple critical state, in the external cycle. The constraints on b are adapted to the ranges $(0, Q_1)$ or $(0, Q_2)$ and the possible values of the discharge Q in the following manner

$$
\begin{aligned}
&b \in (b_1, b_0) \text{ when } \beta < \beta_* \\
&b \in (b_0, b_2) \text{ when } \beta > \beta_* \\
&b_0 = \mathrm{sn}^2(\beta K', k') / \mathrm{cn}^2(\beta K', k') \\
&K' = F(\pi / 2, k')
\end{aligned}
\tag{3.4}
$$

Here, b_0, b_1 and b_2 are the values of this parameter when $Q = 0$, Q_1 and Q_2, respectively, and sn is a Jacobi elliptic function ([5], formula (120.01)).

The next cycle, which is embedded in the external cycle, is now additional with respect to a simple critical state. The determination of f from the fourth equation of system (3.3) is associated with this cycle. Two internal cycles are directly embedded in this cycle and a subsystem of the first two equations of system (3.3) is solved for the parameters d and g in these internal cycles, subject to the constraints (3.1).

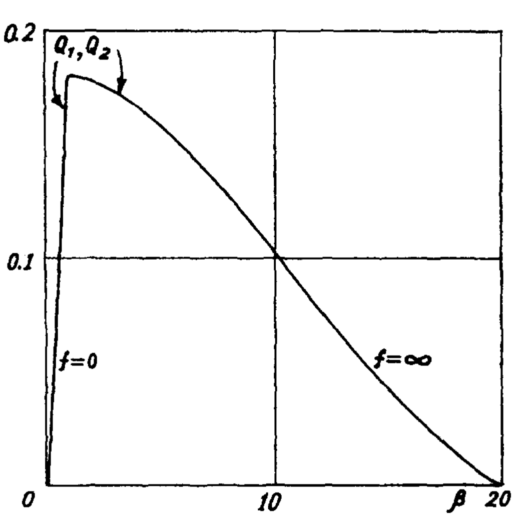

Fig. 3.

As far as the constraints on the parameter f are concerned, their establishment necessitates some additional clarification of certain details.

If it is accepted that the dependences $f(Q)$ and $b(Q)$ are continuous, then, under normal drainage conditions which are close to the critical state obtained with a specified value of b, the relation between the parameters f and b, which is inherent in the latter case, is preserved and, consequently, we have

$$0 < f < b \text{ when } \beta < \beta_*; \quad b < f < \infty \text{ when } \beta > \beta_* \tag{3.5}$$

Returning to Eqs (1.4) and taking the condition $M > 0$ into account, we note that, when the parameters b, d and g are varied when solving system (3.3), each of relations (3.3) between the parameters b and f can be converted into an opposite relation with a change in the sign of the function $J(\rho, b, d, g)$. Its dependence on each of the mapping parameters occurring in it is determined by the inequalities

$$\frac{\partial J}{\partial b} > 0, \quad \frac{\partial J}{\partial g} > 0, \quad \frac{\partial J}{\partial d} < 0 \tag{3.6}$$

which are obtained by differentiating the function Λ_0 [5, formulae (710.11), (730.04)] taking account of expression (1.3) for the modulus k.

Relations (3.6) enable one to identify ranges of values of the parameter b within the framework of which the function J retains its sign for any values of the parameters d and g which are subject to constraints (3.1)

$$J < 0 \text{ when } 0 < b < b_*, \quad J > 0 \text{ when } b_0 < b < \infty; \quad b_* < b_0 \tag{3.7}$$

Here, b_0 is the root of Eq. (2.3) when $d = d_0, g = g_0$, and the value of $b = b_*$ is found from the same equation when $d = d_*, g = g_*$.

In these cases when the initial range of variation of the parameter b specified by the first or second (depending on the relation between the quantities β and β_*) of constraints (3.4) is fitted into the corresponding interval of (3.7), the regulation of (3.5) is preserved for all permissible values of the parameters b, d and g. Taking this into account, the fourth equation of system (3.3) is solved with a fixed parameter b and with the parameters d and g, which are computed in advance and in the internal cycles. The unique solvability of the equation is ensured by the monotonic decrease in the function $F_4(f)$ as the parameters f and b come closer together, which is established in the calculations, and by the fact that the specified values of Q/H_0 is contained within the range of variation of this function. The asymptotic form [2, formula (13.9)]

$$f \approx b + \frac{2K\left(\sqrt{1/d}\right)}{\pi\rho L}\sqrt{\frac{b(1+b)(b+d)}{d}}J(\rho, b, d, g)Q$$

holds for small discharges Q which have a hydrodynamic colour: "extinguishing" of the drain B when the point F, which has a finite velocity, becomes coincident with it.

Additional procedures are provided when there is mismatch of the corresponding ranges (3.4) and (3.7) of variation of the parameter b, which ensures the determination of all of the mapping parameters.

The flow scheme together with the boundary-value problem which describes it are simplified when the surface becomes completely inundated. The critical state of drainage, which is uniquely possible now for any depth of the drain β, is associated with the sole remaining free boundary, that is, the line of separation between the edge and the saline water. The value of Q, which is calculated for such a state, and the range of possible changes in the discharges which is subsequently calculated for other drainage conditions, increases without limit as the drain approaches the inundated surface if, of course, one starts out from the fact that the inflow from the surface is capable of compensating for any outflow from the edge. In this case, it is possible to associate the hydrodynamic flow model with the well-known problem of the outflow of fresh water which is drawn into the ground from so-called infiltration basins and cleansed from impurities during filtration on its way to the water supply.

We will illustrate some special features of the flow at an edge using the results of calculations carried out for $l = 40, L = 50, H_0 = 50, \beta = 2, \rho = 0.02$. Any length scale can be assigned to the input and the calculated geometrical quantities.

The basic results of calculations for double critical drainage conditions are contained in the first line of

Table 1

b	d	g	f	Q	H_1	H_2	y_A
2.4×10^{-3}	1.39	0.973		1.54	27.2	57.7	5.40
0.12×10^{-3}	1.94	0.940	0	0.456	47.3	52.3	1.91
4.92×10^{-3}	1.995	0.932	4.91×10^{-3}	0.0456	49.8	50.2	0.136

Table 1. The parameters d_* and g_* are determined from the system of Eqs (2.4) with constraints (2.5) and the values of $d_0 = 2$ and $g_0 = 0.931$ at which they are calculated using formulae (2.5). The value of the parameter b_* in the same line is connected with the values of d_* and g_* by relation (1.3), which has been noted in Section 2. After the parameters b_*, d_* and g_* have been found, the depth $\beta_* = 5.70$ of the drain and its filtration discharge rate $Q_* = 1.54$, which is the greatest possible discharge rate in this version, are calculated. Three geometrical quantities H_1, H_2 and y_A, which determine the position of the end points of the free boundaries of the edge (Fig. 2a), are included in Table 1. The coordinates of the individual intermediate points of the free boundaries are also calculated for each set of conditions.

In conformity with the choice of the quantity $\beta = 2 < \beta_*$, the calculations at the following stage are carried out for the simple critical state when $f = 0$, associated with the depression curve, point A of which remains a cusp, and are represented by the second line of Table 1. The range of its productivity is reduced considerably when the drain is located higher than in the double critical state. At the same time, the stress in the line of separation is significantly reduced and the upper point C of this line goes off far below its position in the double critical state, approaching the unperturbed level H_0 of the saline ground water. The mapping parameters are determined from the system of Eqs (3.2) with constraints (3.1), the first of which holds the parameter b in the given case.

Under normal drainage conditions, with which the third line in Table 1 is associated, the discharge rate Q is specified to be equal to one tenth of its maximum possible value, when $\beta = 2$, of $Q_1 = 0.456$ which is almost 34 times smaller than the value of Q_* in the double critical state. As a result of this reduction in the drainage, the two free boundaries of the edge are significantly flattened out, as may be judged from the values of H_1, H_2 and y_A. Now, however, the two points of inflection $R_1(23.3; 0.0554)$ and $R_2(1.86; 0.125)$ remain in the depression curve. As in a simple critical state, the separation line has a unique point of inflection $R(24.2; 50.0)$. The tendency noted above for the parameters b and f to approach one another for small values of Q is clearly seen.

4. CALCULATION OF THE SOIL DESALTING ZONE

We shall assume that fresh water begins to enter from the surface into the soil, which initially contains saline ground water, with the simultaneous commencement of drainage with a certain intensity Q. Initially, only saline water drains off through it from the soil, and then, at a certain instant, fresh water, which gradually replaces the saline water.

If the dynamic equilibrium between the filtration flow and the atmosphere is not disturbed during this process, then, in the limit, a steady flow of fresh water is formed above the remaining undisplaced saline water in one of the critical states described above associated with the separation line. The mean thickness H_0 of the layer of fresh water at the edge which has been formed is determined by relation (1.10).

A flattening out of the separation line occurs when the drainage intensity is reduced and with the postulated conservation of the volume of the saline water. An increase in the drainage discharge rate Q disturbs the equilibrium created between the fresh and saline water, leading to additional drainage of the latter and to an increase in the depth H_0 of the demineralization zone. In this case, in order to avoid the breakthrough of air into the drain, the value of Q must not exceed the value Q_* at which its depression curve also turns out to be at the edge which has been formed on the brink of destabilization.

It is therefore also necessary here, at the first stage, to calculate the double critical characteristics, but for a specified ordinate β of the drain. In accordance with this, instead of system (2.4), the system

$$F_1(g) = f_1(b,g) = l / L, \quad F_2(b) = f_2[b, g(b)] = \beta / l \tag{4.1}$$

is now solved.

The parameter d, which is also contained in the functions f_1 and f_2, is computed in advance from relation (2.3) for each pair of parameters b and g, touched upon when solving system (4.1).

After the parameters b, d and g have been found, the intensity of the drainage Q_*, at which a demineralization zone is formed under the stated conditions, is calculated as well as the greatest attainable mean thickness H_0 of the zone in the case of the selected values of the quantities l, L, β and ρ.

Next, the edge in the critical state $f = \infty$, associated with the separation line, is calculated for a certain fixed value of $Q \in (0, Q_*)$. Here, the following system of equations

$$\begin{aligned} F_1(g) &= f_1(b,d,g) = l / L \\ F_2(b) &= f_2[b,d,g(b,d)] = \beta / l \\ F_3(d) &= f_3[b(d),d,g(d)] = Q / L \end{aligned} \tag{4.2}$$

has to be solved.

The third equation in the parameter d, where now $d \in (1, d_*)$, is solved in the external cycle of this procedure. The monotonic increase in each of the functions F_1, F_2, F_3 with respect to its parameter is established numerically and made use of when solving the system. The range of variation in the parameters is controlled by satisfying the relation $J > 0$ which, in this case, ensures that the condition $M > 0$ is satisfied by virtue of (1.4).

We will now illustrate the above using the example of numerical calculations with $l = 40$, $L = 50$, $\beta = 2$, $\rho = 0.02$. The values $Q_* = 0.0440$ and $H_{0*} = 30.4$ are obtained with this combination of input parameters. It is then calculated that $H_0 = 10.3$ and 29.0 for the two specified values of the drainage discharge rates $Q = 0.0440$ and $Q = 0.396$ respectively.

On comparing the values of the quantities Q_*, H_{0*} and β with the values $Q_* = 1.54$, $H_0 = 50$ and $\beta_* = 5.70$ for the numerical example of the preceding problem, found for the same input parameters l, L and ρ in the double critical state, we note an extension of drainage possibilities as a soil demineralizer, with its almost three-fold deepening.

It is interesting that, in the flow model which has been adopted, these possibilities become unlimited regardless of the depth at which the drains are laid in the case of complete inundation of the surface when there is no depression curve which limits the intensification of the drainage. Of course, the discharge rates of real drains, including vacuum drains, is always limited by the special features of their construction, by the water conduction of the soils and the surface water sources which compensate for the drainage. However, the function of the latter in the process being modelled solely consists of creating a layer of fresh ground water of a certain capacity in a deep well zone of the soil, after which the drains can be turned off altogether.

The water improvement problem, with which the modification of the initial boundary-value problem proposed here is associated, was formulated by S. A. Aver'yanov [7]. An approximate solution of the problem was suggested by him, based on replacement of the two free boundaries by fixed horizontal boundaries, subsequent correction of which is made by means of the first boundary conditions (1.1) for these boundaries.

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